DIFFRACTION OF AN ARBITRARY ACOUSTIC WAVE BY A WEDGE

(DIFRAKTSIIA PROIZVOL'NOI AKUSTICHESKOI VOLNY NA KLINE)

PMM Vol.28, № 2, 1964, pp.305-318

A. F. FILIPPOV
(Moscow)

(Received November 23, 1963)

In the problem of diffraction of a wave of arbitrary form by a wedge, it is shown how to obtain any number of terms of the geometric acoustical expansion of the diffracted wave near its front from the known solution of the problem of diffraction of a plane wave by the same wedge. The method which is set forth provides the exact solution in the entire region for certain problems of diffraction of cylindrical and spherical waves.

1. We consider the two-dimensional problem of diffraction of a wave of arbitrary form with curved or straight front by an obstacle in the form of an angle (wedge). The wave propagation is described by Equation

$$U_{tt} = U_{xx} + U_{yy} \tag{1.1}$$

in the region $0 < \varphi < \alpha$, where $x = r\cos\varphi,\ y = r\sin\varphi,\ 0 < \alpha \leqslant 2\pi$. For $t < 0$ the function $U(t,x,y)$ is specified (the incident wave). On the faces of the wedge, $\varphi = 0$ and $\varphi = \alpha$, boundary conditions of the three types

$$\text{(a)}\quad U = 0, \qquad \text{(b)}\quad \frac{\partial U}{\partial n} = 0, \quad \text{(c)}\ \frac{\partial U}{\partial n} = c\frac{\partial U}{\partial t} \qquad (c \geqslant 0) \tag{1.2}$$

are given, where $\partial/\partial n$ is the derivative in the direction of the inner normal to the boundary. We do not exclude cases in which one of conditions (1.2) is given on one face and a different one on the other face. When the incident wave is plane, the exact solution of this problem is known [1 and 2]. The problem of diffraction for any boundary condition of the form

$$(u) \equiv \sum_{p+q+r=n} g_{pqr} \frac{\partial^n U}{\partial t^p \partial x^q \partial y^r} = 0 \tag{1.3}$$

can be solved by this same method if the solution for the problem of diffraction of a plane wave with the same boundary conditions has previously been found (e.g. by the method of [2]).

Let the front of the incident wave MN reach the vertex of the angle at the instant $t = 0$ and let the ray which strikes this point make an angle β with the Ox-axis (Fig. 1); $U = 0$ ahead of the front. The wave fronts are shown in Fig. 1 in the case when $\pi/2 < \beta < \alpha - \pi$. All the results remain valid for other relations between α and β. It is only necessary to take into account that the number and location of the reflected waves will then be different.

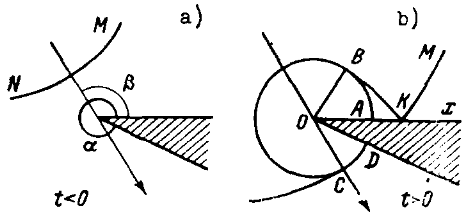

Fig. 1

2. We represent the solution U for $t > 0$ in the form $U = u + v + w$. The function u is equal to zero inside the angle COD (Fig. 1b) and takes on the same values outside this angle which the incident wave would have were there no obstacle. The function v is nonzero only in the region OBK and takes on the same values there that the reflected wave would have if the entire straight line $y = 0$ were the boundary instead of the angle. The reflected wave is found by well-known methods (see below Section 5).

The function w is the diffracted wave which is to be found. Clearly, $w \neq 0$ only in the circle $ABCD$.

In order that the function $U = u + v + w$ be continuous and represent the solution of the problem which has been stated, the function w must satisfy the following requirements. In each of the sectors AOB, BOC and COD it is bounded and satisfies Equation (1.1); on the sides of the wedge OA and OD w satisfies the boundary conditions (a)(1.2), (b)(1.2), or (c)(1.2); on the arcs AB, BC and CD we have $w = 0$; on the radii OC and OB the discontinuities of the function w and of its derivative in the direction normal to the radius are equal in magnitude and opposite in sign to the discontinuities of the function $u + v$ and of its derivative, i.e.

$$[w]_{\pi+\beta} = \mu_0(t, r), \qquad [w]_{\pi-\beta} = \mu_1(t, r)$$
$$\left[\frac{\partial w}{\partial n}\right]_{\pi+\beta} = \nu_0(t, r), \qquad \left[\frac{\partial w}{\partial n}\right]_{\pi-\beta} = \nu_1(t, r), \qquad \frac{\partial}{\partial n} \equiv \frac{1}{r}\frac{\partial}{\partial \varphi} \tag{2.1}$$

where μ_0, μ_1, ν_0 and ν_1 are known functions defined for $0 < r < t$; the the symbol [ ] denotes the magnitude of a discontinuity, for example

$$[w]_{\pi+\beta} \equiv w(t, r, \pi + \beta + 0) - w(t, r, \pi + \beta - 0)$$

3. Let $U^*(t,x,y)$ be the solution of the problem of diffraction of the plane wave which is specified for $t < 0$ by Formulas

$$U^*(t, x, y) = \begin{cases} 0 & \text{for} \quad t \leqslant -x\cos\beta - y\sin\beta \\ 1 & \text{for} \quad t > -x\cos\beta - y\sin\beta \end{cases} \tag{3.1}$$

The solution was obtained in [1] for the case of the boundary conditions (a)(1.2) and (b)(1.2) and in [2] for the case (c)(1.2). The specific form of this solution is not needed at this point. Let $f_0(\tau)$ be a function which is equal to zero for $\tau < 0$.

We set

$$f_i(\tau) = \int_0^\tau \frac{(\tau - s)^{i-1}}{\Gamma(i)} f_0(s)\, ds \qquad (i > 0), \qquad \Gamma(i) \text{ is a gamma function} \tag{3.2}$$

$$U_i^\circ(t, x, y) = \frac{\partial}{\partial t} \int_0^\infty U^*(t - s, x, y) f_i(s)\, ds \qquad (i = 0, 1, 2, \ldots) \tag{3.3}$$

Then

$$f'_{i+1}(\tau) = f_i(\tau) \qquad \text{for } i \geqslant 0$$

$$U_i^\circ(t, x, y) = f_i(t + x\cos\beta + y\sin\beta) \qquad (t < 0) \tag{3.4}$$

By virtue of (3.3) each of the functions U_i° satidfies Equation (1.1) and the same boundary conditions as the function U^*. And so, U_i° is the solution of the problem of diffraction of the plane wave of the form (3.4).

As in Section 2, we shall represent U_i° in the form

$$U_i^\circ = u_i^\circ + v_i^\circ + w_i^\circ \qquad \text{for } t > 0$$

Then

$$[w_i^\circ]_{\pi+\beta} = f_i(t - r), \qquad [\partial w_i^\circ / \partial n]_{\pi+\beta} = 0$$

Taking $f_0(\tau) = \tau^n/n!$ we obtain functions w with discontinuities of the form $c_{nc}\tau^n$, where $\tau = t - r$.

In order to obtain functions w with the discontinuities $c_{lk}\tau^l r^k$, we differentiate U_i° with respect to the angle β. Considering that $f_i' = f_{i-1}$, we obtain from (3.4)

$$\frac{\partial U_i^\circ}{\partial\beta} = -\eta f_{i-1}(t + \xi), \qquad \frac{\partial^2 U_i^\circ}{\partial\beta^2} = -\xi f_{i-1}(t + \xi) + \eta^2 f_{i-2}(t + \xi) \tag{3.5}$$

for $t < 0$, where

$$\xi = x\cos\beta + y\sin\beta, \qquad \eta = -\frac{\partial\xi}{\partial\beta} = x\sin\beta - y\cos\beta, \qquad \frac{\partial\eta}{\partial\beta} = \xi$$

so that for any function u

$$\frac{\partial u}{\partial\beta} = \xi\frac{\partial u}{\partial\eta} - \eta\frac{\partial u}{\partial\xi} \tag{3.6}$$

We introduce the notation

$$\Lambda_{2s-1} = \frac{\partial}{\partial\beta}\left(\frac{\partial^2}{\partial\beta^2} + 1^2\right)\left(\frac{\partial^2}{\partial\beta^2} + 2^2\right) \cdots \left(\frac{\partial^2}{\partial\beta^2} + (s-1)^2\right), \qquad \Lambda_{2s} = \frac{\partial}{\partial\beta}\Lambda_{2s-1} \tag{3.7}$$

$$U_i^m = \Lambda_m U_i^\circ \tag{3.8}$$

Since the unctions U_i^m are derivatives of the solution U_i° with respect to the parameter β, they satisfy Equation (1.1) and the boundary conditions. For $t < 0$ they are the waves with plane fronts obtained from (3.4) in accordance with Formula (3.8). As in Section 2, we set

$$U_i^m = u_i^m + v_i^m + w_i^m$$

In order to compute the discontinuities of the functions w_i^m and of their derivatives on OC it is necessary, according to Section 2, to find the values of U_i^m and $\partial U_i^m/\partial n$ on OC under the assumption that the obstacle (wedge) is absent. The functions U_i^m will then be expressed by Formulas (3.4), (3.7) and (3.8) for $t > 0$ also. It can be seen from (3.5) and (3.6) that $\partial^m U_i^{\circ} / \partial \beta^m$ is an even function of η for even m and an odd function for odd m. Since on OC, i.e. $\varphi = \pi + \beta$, we have $\eta = 0$ and $\partial/\partial n = \partial/\partial \eta$, then

$$U_i^{2k-1}\big|_{\pi+\beta} = 0, \qquad \frac{\partial U_i^{2k}}{\partial n}\bigg|_{\pi+\beta} = 0 \tag{3.9}$$

We shall now prove by induction that

$$U_i^{2k}\big|_{\pi+\beta} = (2k-1)!!\, r^k f_{i-k}(t-r) \qquad ((2k-1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1),\ (-1)!!=1) \tag{3.10}$$

This is true for $k = 1$ in view of (3.8), (3.7) and (3.5) and since $\xi = -r$, $\eta = 0$ on OC. Let us assume that Formula (3.10) is true for some $k \geqslant 1$. We have from (3.7) and (3.8),

$$U_i^{2k+2} = \left(\frac{\partial^2}{\partial \beta^2} + k^2\right) U_i^{2k} \tag{3.11}$$

By virtue of (3.6), we have for any function U

$$\frac{\partial^2 U}{\partial \beta^2} = \xi^2 \frac{\partial^2 U}{\partial \eta^2} - 2\xi\eta \frac{\partial^2 U}{\partial \xi\, \partial \eta} + \eta^2 \frac{\partial^2 U}{\partial \xi^2} - \xi \frac{\partial U}{\partial \xi} - \eta \frac{\partial U}{\partial \eta} \tag{3.12}$$

As a consequence of the invariance of the Laplacian operator under rotation of the coordinate axes, it follows from (1.1) that

$$\partial^2 U / \partial t^2 = \partial^2 U / \partial \xi^2 + \partial^2 U / \partial \eta^2$$

From this result and from (3.12) we obtain on OC, i.e. for $\eta = 0$, $\xi = -r$

$$\frac{\partial^2 U}{\partial \beta^2} = r^2 \left(\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial r^2}\right) - r \frac{\partial U}{\partial r} \tag{3.13}$$

Now we get $$U_i^{2k+2}\big|_{\pi+\beta} = (2k+1)!!\, r^{k+1} f_{i-k-1}(t-r)$$

from (3.11), (3.13) and (3.10).

Formula (3.10) is, therefore, valid for all k.

Since $\partial/\partial n = \partial/\partial \eta = \xi^{-1} \partial/\partial \beta$ on OC, we have from (3.6) to (3.10) that

$$\frac{\partial U_i^{2k-1}}{\partial n}\bigg|_{\pi+\beta} = \frac{1}{\xi} U_i^{2k}\big|_{\pi+\beta} = -(2k-1)!!\, r^{k-1} f_{i-k}(t-r) \tag{3.14}$$

Taking into account what has been said about the discontinuities of the functions w_i^m, we obtain from (3.9), (3.10) and (3.14)

$$[w_{i+k}^{2k}]_{\pi+\beta} = (2k-1)!!\ r^k f_i(t-r), \qquad [w_{i+k}^{2k-1}]_{\pi+\beta} = 0$$
$$\left[\frac{\partial w_{i+k}^{2k}}{\partial n}\right]_{\pi+\beta} = 0, \qquad \left[\frac{\partial w_{i+k}^{2k-1}}{\partial n}\right]_{\pi+\beta} = -(2k-1)!!\ r^{k-1} f_i(t-r) \tag{3.15}$$

4. The diffracted wave w which is sought (see Section 2) will be approximated by the linear combination $w^1 = \Sigma c_{ik} w_i^k$ of the functions w_i^k which were constructed in Section 3. The coefficients c_{ik} are chosen so that the functions w^1 and $\partial w^1/\partial n$ have discontinuities on OC which are equal to the $\mu_0(t,r)$ and $\nu_0(t,r)$ in (2.1) correct to infinitesimals of a specified order in the vicinity of the point $t = 0$, $r = 0$. This can be done if the incident wave is represented by a sufficiently smooth function, since the functions μ_0 and ν_0 then have the Taylor expansion in τ and r $(\tau = t - r)$

$$\mu_0 = u\,|_{\pi+\beta} = \sum_{i=0}^{s}\sum_{k=0}^{s-i} a_{ik}\frac{\tau^{N+i} r^k}{(N+i)!\,k!} + o(\tau^N t^s) \tag{4.1}$$

$$\nu_0 = \frac{\partial u}{\partial n}\Big|_{\pi+\beta} = \sum_{i=0}^{s-1}\sum_{k=0}^{s-i-1} b_{ik}\frac{\tau^{N+i} r^k}{(N+i)!\,k!} + o(\tau^N t^{s-1}) \tag{4.2}$$

We take

$$f_i(\tau) = \tau^{N+i}/(N+i)! \quad (\tau > 0), \qquad f_i(\tau) = 0 \quad (\tau \leqslant 0)$$

By virtue of (3.15) the discontinuities of the function

$$w^1 = \sum_{i=0}^{s}\sum_{k=0}^{s-i}\frac{2^k a_{ik}}{(2k)!} w_{i+k}^{2k} - \sum_{i=0}^{s-1}\sum_{k=0}^{s-i-1}\frac{2^k b_{ik}}{(2k+1)!} w_{i+k+1}^{2k+1} \tag{4.3}$$

and of the derivative $\partial w^1/\partial n$ on OC differ from μ_0 and ν_0 only by $o(\tau^N t^s)$ and $o(\tau^N t^{s-1})$. The discontinuities on OB will also differ only slightly. (For the boundary conditions (a) (1.2) and (b) (1.2) this follows from the well-known law of reflection $v(t, x, y) = \mp u(t, x, -y)$; for case (c) (1.2) it is proved in Section 6). The function $u + v + w^1$ is an approximate solution of the diffraction problem posed in Section 1. It differs from the exact solution $u + v + w$ by the quantity $w^1 - w$, which, as is shown in Section 7, is small in the zone near the front compared to the functions w_i^s which enter into (4.3).

We remark that in the derivation of Formulas (3.15) in Section 3 the derivatives $\partial^l u_{i+k}/\partial\beta^l$, $l \leqslant 2k \leqslant 2s$ were used. These derivatives exist if N is sufficiently large. Therefore, Formula (4.3) is justified at present only in the case when the incident wave can be expanded according to (4.1) and (4.2) with sufficiently large N, $N \geqslant N_s$. If, however, $N < N_s$, let us integrate the incident wave $(N_s - N)$ times with respect to t. The wave which is obtained can be expanded in accordance with (4.1), (4.2) with $N = N_s$. Therefore, the function (4.3) may be written for this wave. We obtain the function w^1 for the incident wave given originally by differentiation $(N_s - N)$ times with respect to t. This proves Formula (4.3) for all $N \geqslant 0$.

If the coefficients c_{ik} in the expression $w^1 = \Sigma c_{ik} w_i^k$ are chosen so that the discontinuities $[w^1]_{\pi+\beta}$ and $[\partial w^1 / \partial n]_{\pi+\beta}$, which are determined with the aid of Formulas (3.15), uniformly approximate the functions μ_0 and ν_0 in (2.1) in the entire region $0 \leqslant r \leqslant t \leqslant t_1$, where t_1 is any constant, then the function w^1 will serve as an approximation to the unknown diffracted wave w in the entire cone $x^2 + y^2 \leqslant t^2 \leqslant t_1^2$ (in the case of the boundary conditions (a) (1.2) and (b), (1.2)). This follows from the estimates of Section 7.

5. We shall now study some properties of the reflected wave. Let the function $U(t,x,y)$ satisfy: Equation (1.1) for $t \geqslant 0$ in the region $y \geqslant 0$, the boundary condition (1.3) for $y = 0$, and the initial conditions $U = \varphi(x,y)$, $U_t = \psi(x,y)$ for $t = 0$. The solution u of Equation (1.1) in the region $-\infty < y < \infty$ having the initial conditions $u = \varphi(x, y)$, $u_t = \psi(x, y)$ for $y \geqslant 0$, and $u = 0$, $u_t = 0$ for $y < 0$ is called the incident wave. The function $v = U - u$ is called the reflected wave.

L e m m a o n t h e s h i f t o f t h e b o u n d a r y . If $v(t,x,y)$ is the wave reflected from the boundary $y = 0$, then for any $h \geqslant 0$ the function $v^* (t, x, y) \equiv v (t, x, y + 2h)$ in the region $y \geqslant -h$ is a wave reflected from the boundary $y = -h$ with the same boundary conditions (1.3) (for the same initial conditions $U = \varphi$, $U_t = \psi$, $t = 0$, the definitions being extended to $\varphi = \psi = 0$ for $y < 0$).

P r o o f . It is easy to see that $v(t,x,y) = 0$ for $0 \leqslant t \leqslant y$ and that for $y \geqslant -h$ the function

$$U^*(t, x, y) = u (t, x, y) + v (t, x, y + 2h)$$

satisfies Equation (1.1) and the boundary conditions indicated in Lemma. We shall show that the function U^* satisfies the boundary condition (1.3) for $y = -h$, i.e. that $N(t,x,h) = 0$, where $N (t, x, h) = l (U^*) |_{y=-h}$. We have

$$\frac{\partial}{\partial h} \{l (v (t, x, y + 2h))_{y=-h}\} = \left\{\frac{\partial}{\partial y} l (v (t, x, y + 2h))\right\}_{y=-h} =$$
$$= \left\{l \left(\frac{\partial}{\partial y} v (t, x, y + 2h) \right)\right\}_{y=-h}$$

Analogous equations hold for $\partial l(u)/\partial h$ and for the second derivatives with respect to h, t and x. Since u and v satisfy Equation (1.1),

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial h^2}\right) N (t, x, h) = 0 \qquad (t \geqslant 0,\ h > 0)$$

Furthermore, $N (t, x, 0) = l (u + v) |_{y=0} = 0$ by virtue of (1.3). Finally, for $t = 0$, $h > 0$ we have $N = 0$, $\partial N / \partial t = 0$ (since $u = 0$ for $0 \leqslant t < - y$, $v (t, x, y) = 0$ for $0 \leqslant t < y$). Thus the function $N(t,x,h)$ satisfies the wave equation and zero initial and boundary conditions. It is, therefore, equal to zero.

C o r o l l a r y . The function $u (t, x, y - h) + v (t, x, y + h)$ satisfies the boundary condition (1.3) for $y = 0$ for any $h \geqslant 0$, i.e.

$$l (u (t, x, y - h) + v (t, x, y + h)) |_{y=0} = 0 \qquad (5.1)$$

We shall now establish the connection between the values of the incident wave on the plane $x \sin\beta - y\cos\beta = 0$ and the reflected wave on the symmetric plane $x \sin\beta + y\cos\beta = 0$ in the case when the boundary condition (1.3) is satisfied for $y = 0$. We introduce new systems of coordinates τ, p, n which are different for the incident and reflected waves. For the incident wave

$$p = -x\cos\beta - y\sin\beta, \qquad n = x\sin\beta - y\cos\beta, \qquad \tau = t - p$$

and for the reflected wave y is replaced by $-y$ in these formulas.

L e m m a . Let us assume that on the plane $n = 0$ Formulas

$$u = \sum_{i=1}^{M}\sum_{j=0}^{M-i} a_{ij}\frac{\tau^i p^j}{i!\,j!} + o(t^M), \qquad \frac{\partial u}{\partial n} = \sum_{i=1}^{M-1}\sum_{j=0}^{M-i-1} b_{ij}\frac{\tau^i p^j}{i!\,j!} + o(t^{M-1}) \tag{5.2}$$

are valid for the incident wave in the vicinity of the origin when $t \geqslant |p|$ and that analogous formulas hold for the reflected wave v, but with the coefficients c_{ij} and d_{ij}. Then $c_{ij}\ (i+j \leqslant M)$ and $d_{ij}\ (i+j \leqslant M-1)$ depend only upon $a_{ij}\ (i+j \leqslant M)$, $b_{ij}\ (i+j \leqslant M-1)$, the angle β and on the coefficients g_{pqr} of the boundary condition (1.3). They do not depend on the other parameters of the incident wave.

P r o o f . We shall carry out the proof for the boundary condition (c) (1.2); in the case (1.3) the arguments are similar. By virtue of (5.1) we have

$$u_y(t, x, -h) + v_y(t, x, h) - cu_t(t, x, -h) - cv_t(t, x, h) = 0 \qquad \text{for } h \geqslant 0$$

Transforming to τ, p, n, we get

$$(v_p - u_p - v_\tau + u_\tau)\sin\beta + (v_n - u_n)\cos\beta - c(u_\tau + v_\tau) = 0 \tag{5.3}$$

where the values of τ, p and n are one and the same for u and v

$$p = h\sin\beta - x\cos\beta, \qquad n = x\sin\beta + h\cos\beta, \qquad \tau = t - p, \qquad h \geqslant 0$$

And so Equation (5.3) is valid in some region of the space τ, p, n which adjoins the origin coordinates. Differentiating (5.3) with respect to n and making use of the fact that u and v satisfy Equation (1.1), i.e. the equation $2U_{\tau p} = U_{pp} + U_{nn}$, we obtain

$$(v_{pn} - u_{pn} - v_{\tau n} + u_{\tau n})\sin\beta + (2v_{\tau p} - 2u_{\tau p} - v_{pp} + u_{pp})\cos\beta - cu_{\tau n} - cv_{\tau n} = 0 \tag{5.4}$$

Substituting (5.2) and (5.3) into (5.4) and comparing coefficients of $\tau^i p^j / i!j!$, we get

$$(c - \sin\beta)\, a_{i+1,\, j} + (c + \sin\beta)\, c_{i+1,\, j} = (c_{i,\, j+1} - a_{i,\, j+1})\sin\beta + (d_{ij} - b_{ij})\cos\beta$$

$$(c - \sin\beta)\, b_{i+1,\, j} + (c + \sin\beta)\, d_{i+1,\, j} = (d_{i,\, j+1} - b_{i,\, j+1})\sin\beta + (a_{i,\, j+2} - c_{i,\, j+2})\cos\beta + 2(c_{i+1,\, j+1} - a_{i+1,\, j+1})\cos\beta$$

where $i = 0, 1, 2, \ldots$, $j = 0, 1, 2\ldots$, with $a_{0j} = b_{0j} = c_{0j} = d_{0j} = 0$. Since $c + \sin\beta \neq 0$, it is possible to find c_{1j}, $j = 0, 1, 2, \ldots$ first from these formulas, then d_{1j}, then c_{2j}, d_{2j}, etc. The assertion is then proved.

N o t e . It can be shown that in the case of the boundary conditions (1.3)

$$c_{0j} = ka_{0j}, \; c_{ij} = \sum_{s=0}^{i} \frac{2^s}{(2s)!} \Lambda_{2s}(k)\, a_{i-s,\, j+s} - \sum_{s=1}^{i} \frac{2^{s-1}}{(2s-1)!} \Lambda_{2s-1}(k)\, b_{i-s,\, j+s-1}$$

$$d_{ij} = \sum_{s=0}^{i} \frac{2^s}{(2s)!} \Lambda_{2s}(k)\, b_{i-s,\, j+s} + \sum_{s=0}^{i} \frac{2^s}{(2s)!}\left(s\Lambda_{2s-1}(k) - \frac{2}{2s+1} \Lambda_{2s+1}(k)\right) a_{i-s,\, j+s+1}$$

where the Λ_s are the same as in (3.7) and $k = k(\beta)$ is the known reflection coefficient for plane waves

$$k(\beta) = -\frac{m(\beta)}{m(-\beta)}, \qquad m(\beta) = \sum_{p+q+r=n} g_{pqr} \cos^q\beta \sin^r\beta \tag{5.5}$$

6. We now estimate the discontinuities of the functions $w^1 - w$ and $\partial(w^1 - w)/\partial n$ on OB in the case of boundary condition (c) (1.2). Let the functions $f_1(\tau)$, w^1, w be the same as in Section 4 and U^1 be the sum of the solutions $U_1{}^k$ constructed in Section 3 with the same coefficients as in Formula (4.3). Then, as in Section 2, $U^1 = u^1 + v^1 + w^1$, where u^1 is an incident wave, v^1 the reflected wave, and w^1 the diffracted wave. The function w^1 here is obviously the same as in (4.3). Since the function U^1, the exact solution $U = u + v + w$, and the incident waves u and u^1 are continuous on OB, i.e. for $\varphi = \pi - \beta$, then $[w^1 - w]_{\pi-\beta} = -[v^1 - v]_{\pi-\beta}$. The same conclusion can also be drawn concerning the derivatives $\partial/\partial n$. The coefficients in Formula (4.3) have been chosen so that the functions u^1 and $\partial u^1/\partial n$ on OC have the same coefficients in a power series expansion in τ and r as u and $\partial u/\partial n$ have in (4.1) and (4.2). Then according to the second lemma of Section 5, the same holds for v^1, $\partial v^1/\partial n$ and v, $\partial v/\partial n$ on OB; here $M = N + s$. Therefore

$$v^1 - v\,|_{\pi-\beta} = o(t^M), \qquad \partial(v^1 - v)/\partial n\,|_{\pi-\beta} = o(t^{M-1})$$

From this it follows that

$$[w^1 - w]_{\pi-\beta} = o(t^M), \qquad \left[\frac{\partial(w^1 - w)}{\partial n}\right]_{\pi-\beta} = o(t^{M-1}) \tag{6.1}$$

To justify the applicability of this lemma it must be shown that the reflected wave exists and admits an expansion analogous to (5.2). This follows from [4] for a sufficiently smooth incident wave. It is even possible to obtain an explicit solution by the method which Sobolev used to find the solution of the problem of propagation of elastic waves for the case of the half plane [1], pp. 509-559.

7. We now estimate the difference $w^1 - w$. It follows from the preceding that the function $w^1 - w$ satisfies Equation (1.1) in the region $r < t$ excepting the planes $\varphi = \pi - \beta$ and $\varphi = \pi + \beta$, which are represented by the lines OB and OC in Fig. 1, where the function and its derivatives have the discontinuities

$$[w^1 - w] = \mu_i{}^*(t, r) = o(t^M), \qquad \left[\frac{\partial(w^1 - w)}{\partial n}\right] = \nu_i{}^*(t, r) = o(t^{M-1})$$

for OC $i = 0$, for OB $i = 1$. Further, for $r = t$ the function $w^1 - w$ equals zero and for $\varphi = 0$ and $\varphi = \alpha$ satisfies the boundary conditions (a) (1.2), (b) (1.2), or (c) (1.2). We shall consider that the functions $\mu_i{}^*$ and $\nu_i{}^*$ have derivatives up to the fourth order, the kth derivatives of $\mu_i{}^*$ being equal to $o(t^{M-k})$, $k \leqslant 4$, and of $\nu_i{}^* - o(t^{M-k-1})$ (this occurs if the incident wave is sufficiently smooth). Let us replace t by $\tau + r$ and expand μ_i^* and ν_i^* according to Taylor's formula

$$\mu_i^* = \mu_i^\circ(\tau) + r\mu_i^1(\tau) + r^2\mu_i^2(\tau, r), \qquad \nu_i^* = \nu_i^\circ(\tau) + r\nu_i^1(\tau, r)$$

Obviously

$$\mu_i^\circ = o(\tau^M), \quad \mu_i^1 = o(\tau^{M-1}), \quad \nu_i^\circ = o(\tau^{M-1})$$
$$\mu_i^2 = o(t^{M-2}), \quad \nu_i^1 = o(t^{M-2}), \quad \text{for } t \to 0$$

In the cases of boundary conditions (a) (1.2) and (b) (1.2), let

$$z^* = w^1 - w - W$$

$$W = \frac{\partial^N}{\partial t^N}\left\{\sum_{j=0}^{1}\int_0^{t-r} w_j^{2j}(t-s, x, y)\, d\mu_1^j(s) - \int_0^{t-r} w_1^1(t-s, x, y)\, d\nu_1^\circ(s)\right\}$$

where $w_1{}^k$ is the same as in Section 3, $r^2 = x^2 + y^2$. Then

$$[z^*]_{\pi+\beta} = r^2\mu_0^2(\tau, r), \qquad [\partial z^* / \partial n]_{\pi+\beta} = r\nu_0^1(\tau, r)$$

it is easy to see that the discontinuities on OB are also equal to

$$r^2 o\left(t^{M-2}\right) \quad \text{and} \quad r o\left(t^{M-2}\right).$$

In the case of boundary condition (c) (1.2) it is also possible to construct a function z^* with discontinuities of the same order, but to do this it is necessary to add terms with discontinuities on only OB to the function W (we shall not dwell on this point in greater detail).

Let us take a small $h > 0$ and construct a function z which coincides with z^* outside the sectors S_i, defined by the inequalities

$$|\varphi - \beta_i| < h, \qquad r < t, \qquad (i = 0, 1;\ \beta_0 = \pi + \beta,\ \beta_1 = \pi - \beta)$$

and which is continuous together with the derivative $\partial z/\partial\varphi$ in these sectors, including their lateral boundaries

$$z = z^* + \left(\frac{\mu_i^2}{2h^2}\operatorname{sign}(\beta_i - \varphi) + \frac{r\nu_i^1}{4h}\right)(h - |\varphi - \beta_i|)^2 \qquad \text{for } |\varphi - \beta_i| \leqslant h$$

Then z satisfies the same boundary conditions as $w^1 - w$ and Equation

$$z_{tt} - z_{xx} - z_{yy} = f(t, x, y) \tag{7.1}$$

Here $f = 0$ outside S_i, $f = h^{-2}o(t^{M-2})$ in S_i. The last estimate follows from the properties of μ_i, ν_i and their derivatives. We now estimate the "energy"

$$E(t) = \iint_D \frac{1}{2}(z_t^2 + z_x^2 + z_y^2)\, dx\, dy$$

of the solution z at the instant $t = t_1 > 0$. Integrating by parts and taking (7.1) into consideration, we obtain

$$\frac{dE}{dt} = \iint_D z_t f\, dx\, dy - \int_\Gamma z_t z_n\, ds$$

where Γ is the boundary of the region $D(0 < \varphi < \alpha)$ traversed in the positive direction, and z_n is the derivative in the direction of the inner normal to Γ; we note that $z = 0$ for $r > t$. In the case of the boundary condition (c) (1.2) we have $z_n = \sigma z_t$, so that the contour irtegral $\geqslant 0$, and in the cases (a) (1.2) and (b) (1.2) it equals zero. Therefore,

$$\frac{dE}{dt} \leqslant \iint z_t f\, dx\, dy$$

Making use of Schwartz's inequality, we obtain

$$\frac{dE}{dt} \leqslant \left(\iint z_t^2 dx\,dy \iint f^2\,dx\,dy\right)^{1/2} \leqslant \left(2E\,(t) \iint f^2\,dx\,dy\right)^{1/2}$$

$$\frac{1}{\sqrt{E\,(t)}}\,\frac{dE}{dt} \leqslant \left(2\iint f^2\,dx\,dy\right)^{1/2} \leqslant \gamma\,(t)\,t^{M-1}, \qquad \gamma\,(t) \to 0 \qquad (t \to 0)$$

Intergrating and taking into account that $E(0) = 0$, we get

$$E\,(t_1) \leqslant \frac{4}{M^2}\,t_1^{\,2M} \max_{0 \leqslant t \leqslant t_1} \gamma^2\,(t) = o(t_1^{2M})$$

Let us estimate z near the front $r = t$ for $t > t_1$. Let Z_τ be the zone at the front given by

$$t - \tau < r < t, \qquad 0 < \varphi < \alpha, \qquad |\varphi - \beta_i| > h + \psi \qquad (i = 0, 1), \qquad \psi > 0$$
$$\tau = t_1\,(1 - \cos\psi).$$

It is easy to see that because of the finiteness of the velocity of propagation of the waves, values of the function f in the sectors S_1 for $t > t_1$ do not affect the values of z in the zone Z_τ. Since $f = 0$ outside these sectors, the energy included in the zone Z_τ at the instant t , i.e.

$$\iint\limits_{Z_\tau} \frac{1}{2}\,(z_t^{\,2} + z_x^{\,2} + z_y^{\,2})\,dx\,dy$$

does not exceed the entire energy which was present at the instant t_1, i.e. $E(t_1)$. Thus, the energy of the solution z in the zone Z_τ near the front equals $o\,(\tau^{2M})$.

In this zone $w_i^{\,k} = O\,(\tau^{N+i+1/2}r^{-1/2})$. Using the corresponding estimates for the derivatives, we may conclude that the energy of the solution W is also equal to $o\,(\tau^{2M})$. Thus, the energy of the solution $w^1 - w$ is equal to $o\,(\tau^{2M})$. in this zone.

In Formula (4.3) Equations $w_s^{\,j}$ were of the highest order of small quantities near the front; in the zone Z_τ

$$w_s^{\,j} = g_s^{\,j}\,(\tau, r, \varphi)\,\tau^{N+s+1/2}r^{-1/2}$$

where $g_s^{\,j}$ is a smooth function. Therefore, their energy in the zone Z_τ is equal to $O\,(\tau^{2M})$. Thus, the error of Formula (4.3), i.e. $w^1 - w$, is an infinitesimal of higher order compared to any term of this formula in the zone Z_τ at the front as $\tau \to 0$.

8. In the case where the front of the incident wave is a circular arc convenient formulas can be derived for any number of terms of the geometric acoustical expansion of the diffracted wave. (This case is the most important one, since it permits consideration of the diffraction of a wave due to a point source, and also of multiple diffraction by the vertices of a polygon,* by a segment and by a slit).

We shall now indicate some properties of the geometric acoustical expansion of an arbitrary wave with a circular front. For $t > t_0$ let the front be an arc of the circle $\rho = t - t_0$ (in polar coordinates ρ, θ), and let the wave itself be specified by Formulas

---

* By a simpler method than this was done in [6].

$$u = 0 \quad \text{for } \tau < 0, \qquad u = \sum_{i=0}^{m} f_i(\tau) A_i(\rho, \theta) + \overset{*}{A}_{m+1}(\tau, \rho, \theta) \quad \text{for } \tau \geqslant 0 \qquad (8.1)$$

Here the f_i are the same as in (3.2),

$$\tau = t - t_0 - \rho, \qquad |\overset{*}{A}_{m+1}| \leqslant C \int_0^{\tau} |f_m(s)|\, ds$$

Then (see [5], Formulas (27) and (34), where to transform to the notation of this paper φ must be replaced by u, q by ρ, $|q|^{-\frac{1}{2}} A_i$ by A_i).

$$2 \frac{\partial A_i}{\partial \rho} + \frac{1}{\rho} A_i = \frac{\partial^2 A_{i-1}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A_{i-1}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 A_{i-1}}{\partial \theta^2}$$

From this,

$$A_0 = \frac{1}{\rho^{1/2}} a(\theta), \qquad A_1 = -\frac{1}{2\rho^{3/2}} \left(\frac{\partial^2 a}{\partial \theta^2} + \frac{1}{2^2} a \right) + \frac{b(\theta)}{\rho^{1/2}}, \ \ldots$$

where $a(\theta)$, $b(\theta)$, ... are arbitrary functions of θ. First we examine the case in which these functions, beginning with $b(\theta)$, are equal to zero. We then obtain

$$A_i = \frac{(-1)^i L_{2i}^{\theta} a(\theta)}{2^i i!\, \rho^{i+1/2}}, \qquad L_{2i}^{\theta} \equiv \left(\frac{\partial^2}{\partial \theta^2} + \left(\frac{1}{2}\right)^2 \right) \left(\frac{\partial^2}{\partial \theta^2} + \left(\frac{3}{2}\right)^2 \right) \cdots \left(\frac{\partial^2}{\partial \theta^2} + \left(i - \frac{1}{2}\right)^2 \right)$$

If the function $a(\theta)$ is analytic, the series

$$\sum_{i=0}^{\infty} \frac{(-1)^i L_{2i}^{\theta} a(\theta)}{2^i i!\, \rho^{i+1/2}} f_i(\tau) = F(\rho, f_0(\tau), a(\theta)) \qquad (8.2)$$

converges for $0 < \tau/\rho < \gamma_0$, in which γ_0 depends only on the radius of convergence of the power series for $a(\theta)$.

We now consider the case in which $b(\theta)$, $c(\theta)$, ... may also be nonzero.

Any wave (8.1) with a circular wave front may then be represented in the form

$$u = F(\rho, f_0(\tau), a(\theta)) + F(\rho, f_1(\tau), b(\theta)) + F(\rho, f_2(\tau), c(\theta)) + \ldots + R_{m+1}$$

where the remainder term R_{m+1} is of the same order as A^*_{m+1} in (8.1). It suffices, therefore, to examine the problem of diffraction of a wave of the form (8.2).

Let the angle which gives rise to the diffraction be the same as in Fig.1, and let r, φ be polar coordinates with pole at the vertex of the angle, O. The center of the incident circular wave is the point O_0 ($r = R$, $\varphi = \beta$), and ρ and θ are polar coordinates with pole O_0. The incident wave is assumed to have the form (8.2), where

$$f_i(\tau) = \tau^{N+i} / \Gamma(N + i + 1) \quad \text{for } \tau > 0$$

In order to find the diffracted wave it is necessary (see Section 4) to expand the functions u and $\partial u/\partial n$ on OC in Taylor series in r (we have

$r = \rho - R$, $\partial u/\partial \imath = \rho^{-1}\partial u/\partial\theta$ on OC) and to substitute into (4.3) the coefficients obtained from (4.1) and (4.2),

$$a_{ik} = \frac{(-1)^{i+k}(2i+2k-1)!!\,L_{2i}^{\theta}a}{2^k(2i)!\,R^{i+k+1/2}}, \qquad b_{ik} = \frac{(-1)^{i+k}(2i+2k+1)!!\,L_{2i+1}^{\theta}a}{2^k(2i+1)!\,R^{i+k+3/2}}$$

$$\left(L_{2i+1}^{\theta} \equiv \frac{\partial}{\partial\theta}L_{2i}^{\theta}\right)$$

after setting $\theta = \pi + \beta$.

We shall now write out the geometric acoustical expansion of the functions $w_p^{\,1}$ in (4.3). The solution $U^*(t,x,y)$ in Section 3 is a homogeneous function of degree zero.

If $f_i(\tau) = \tau^{N+i}/\Gamma(N+i+1)$ for $\tau > 0$, then the functions $U_p^{\,1}$ in (3.3) and (3.8), and so also the diffracted waves $w_p^{\,1}$, are of degree $N+p$. They can, therefore, be expressed by formulas of the form (8.2)

$$w_p^{\circ} = \sum_{i=0}^{\infty}\frac{(-1)^i L_{2i}^{\varphi}m(\varphi,\beta)}{2^i i!\,r^{i+1/2}}\;\frac{\tau^{N+p+i+1/2}}{\Gamma(N+p+i+3/2)}, \qquad w_p^{\,l} = \Lambda_l w_p^{\circ} \qquad (8.3)$$

We find from [1] that in the case of boundary condition (a) (1.2)

$$m = \frac{1}{2\alpha}\sqrt{\frac{\pi}{2}}\sum_{k=1}^{4}(-1)^k \operatorname{ctg}\frac{\pi(\varphi-\gamma_k)}{2\alpha}, \qquad \begin{array}{l}\gamma_1 = -\gamma_3 = \pi+\beta\\ \gamma_2 = -\gamma_4 = \pi-\beta\end{array}$$

for the case (b) (1.2) it is only necessary to change the sign of the first and fourth terms of the sum. For the case of boundary condition (c) (1.2) the function m can be determined from [2].

Substituting a_{ik}, b_{ik}, $w_p^{\,1}$ into (4.3) and setting $s = \infty$ (using well-known estimates for the derivatives of analytic functions, the absolute convergence of all the series obtained can be proved for sufficiently small τ/r and τ/R), we obtain

$$w^1 = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l=0}^{2p}\frac{(-1)^{p+l}C_{2p}^{\,l}\,(L_l^{\beta}a(\pi+\beta))\,(\Lambda_{2p-l}^{\beta}M_j)}{2^p p!\,R^{p+1/2}}\; f_{p+j+1/2}(\tau)$$

$$M_j = \frac{(-1)^j}{2^j j!}L_{2j}^{\varphi}m(\varphi,\beta), \qquad C_{2p}^{l} = \frac{(2p)!}{l!(2p-l)!}$$

Noting that for any functions a and b

$$\sum_{l=0}^{2p} C_{2p}^{\,l}\,(L_l a)\,(\Lambda_{2p-l}b) \equiv L_{2p}(ab)$$

(this identity can be proved by induction on p), we obtain

$$w^1 = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n L_{2n-2j}^{\beta}\,(a(\pi+\beta)\,L_{2j}^{\varphi}m(\varphi,\beta))}{2^n j!\,(n-j)!\,R^{n-j+1/2}r^{j+1/2}}\, f_{n+1/2}(\tau) \qquad (8.4)$$

This series converges in some neighborhood of the front of the diffracted wave. The sphere of convergence shrinks to zero as we approach the singular points B and C (Fig. 1).

The formula (8.4) was obtained for an incident wave of the form (8.2), where $f_i(\tau) = \tau^{N+i} / \Gamma(N+i+1)$ for $\tau > 0$. If the solution for this case is denoted by $U_N(t,x,y)$, the solution U for the case of any integrable $f_0(\tau)$ which is equal to zero for $\tau < 0$ is the superposition of the solutions $U_N(t-s, x, y)$

$$U(t, x, y) = \frac{\partial^{N+1}}{\partial t^{N+1}} \int_0^\infty U_N(t-s, x, y) f_0(s)\, ds \tag{8.5}$$

since the incident wave is a superposition of the same kind. From this it is easily found that Formula (8.4) is also valid for such f_0. Here $f_{n+1/2}(\tau)$ in (8.4) is expressed in terms of $f_0(\tau)$ by Formula (3.2).

9. We now apply the results which have been obtained to the problem of diffraction of the wave caused by the simplest point source acting at the point $r = R$, $\varphi = \beta$ at the instant $t = -R$, i.e. the wave

$$u = \frac{1}{2\pi \sqrt{(t+R)^2 - \rho^2}} \qquad (t + R > \rho) \tag{9.1}$$

Here ρ is the distance from the source to the point of observation. Setting $t + R = \rho + \tau$ we obtain

$$u \sim \frac{1}{2\sqrt{2\pi\rho}} \frac{\tau^{-1/2}}{\Gamma(1/2)} \quad \text{for} \quad \tau \to +0$$

Therefore, taking

$$a(\theta) \equiv \frac{1}{2\sqrt{2\pi}}, \qquad f_0(\tau) = \frac{\tau^{-1/2}}{\Gamma(1/2)}, \qquad f_{n+1/2}(\tau) = \frac{\tau^n}{n!}$$

in (8.2) and (8.4), we find the diffracted wave w^1.

In the cases of the boundary conditions (a) (1.2) and (b) (1.2) the formulas which have been obtained for w^1 can be simplified further if it is noted that in these cases $\partial^{2s} m / \partial\beta^{2s} \equiv \partial^{2s} m / \partial\varphi^{2s}$, $s = 1, 2, \ldots$, and if the identity

$$L_{2s} L_{2j} \equiv \sum_{q=0}^{\min(s,\, j)} \frac{(-1)^q\, s!\, j!\, (s+j)!}{q!\,(s-q)!\,(j-q)!\,(s+j-q)!} L_{2s+2j-2q}$$

(which can be proved by induction on j) is used. The summation on j in (8.4) is then carried out with the aid of the binomial theorem and the terms containing L_{2k} with the same k are collected. We obtain

$$w^1 = \frac{1}{2\sqrt{2\pi R}} \sum_{k=0}^\infty \frac{(-1)^k L_{2k} m}{2^k k!\, r^{k+1/2}} \frac{z^k}{k!}, \qquad z = \frac{R+r}{R}\tau + \frac{\tau^2}{2R}$$

From a comparison with (8.3), it follows that

$$w^1(\tau, r, \varphi) = \frac{1}{2\sqrt{2\pi R}} w^\circ(z, r, \varphi)$$

Here $w^\circ(\tau, r, \varphi)$ is the diffracted wave for the incident plane wave

$$u^\circ = \frac{\tau^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi\tau}}$$

which approaches from the same direction β. Transforming from τ to $t = \tau + r$, we obtain

$$w^1(\tau, r, \varphi) = \frac{1}{2\pi\sqrt{2R}} w\left(t + \frac{t^2 - r^2}{2R}, r, \varphi\right) \tag{9.2}$$

Here $w(t, r, \varphi)$ is the diffracted wave for the incident plane wave which is given by Formulas

$$U = \frac{1}{\sqrt{t + r\cos(\varphi - \beta)}} \quad (t > -r\cos(\varphi - \beta)), \quad U = 0 \quad (t < -r\cos(\varphi - \beta)) \tag{9.3}$$

for $t < 0$.

With the aid of (8.5) we get the solution of the problem of the diffraction of the wave (9.3)

$$U(t, r, \varphi) = \frac{\partial}{\partial t}\int_0^\infty U^*(t - s, r, \varphi)\frac{ds}{\sqrt{s}} \tag{9.4}$$

where $U^*(t, r, \varphi)$ is the solution found by Sobolev of the problem of diffraction of the wave (3.1). Replacing the solution U^* in (9.4) by the diffracted wave w^* (as in Section 2, $U^* = u^* + v^* + w^*$), we obtain function $w(t, r, \varphi)$.

Formula (9.2) was derived with the aid of series which converge in some neighborhood of the front $r = t$ and, consequently, at this stage it has been proved only for this neighborhood. We shall now prove that Formula (9.2) is, in fact, valid throughout.

T h e o r e m . If $u^\circ(t, r, \varphi)$ is a homogeneous solution of the wave equation

$$Lu^\circ \equiv u^\circ_{tt} - u^\circ_{rr} - r^{-1}u^\circ_r - r^{-2}u^\circ_{\varphi\varphi} = 0$$

and is of degree $-\frac{1}{2}$ in r and t, then

$$u(t, r, \varphi) = u^\circ\left(t + \frac{t^2 - r^2}{2R}, r, \varphi\right) \tag{9.5}$$

is also a solution of this equation.

P r o o f . We have

$$Lu = \frac{2}{R}\left(Tu^\circ_{TT} + ru^\circ_{Tr} + \frac{3}{2}u^\circ_T\right) = \frac{2}{R}\left(Tv_T + rv_r + \frac{3}{2}v\right)$$

Here

$$T = t + \frac{t^2 - r^2}{2R}, \quad v = u^\circ_T$$

v is a homogeneous function of degree $-\frac{3}{2}$. According to Euler's theorem on homogeneous functions $Tv_T + rv_r \equiv -3v/2$, i.e. $Lu = 0$.

N o t e 1. The theorem is also valid for generalized solutions of the wave equation.

By a generalized solution of the wave equation $Lu = 0$ in the region D we mean a function (or generalized function) u, such that for any v which has derivatives of all orders and is equal to zero in the neighborhood of the boundary D and outside some finite region

$$\iiint_D uLv\, dt\, dx\, dy = 0$$

Let $u^\circ(t,r,\varphi)$ be a homogeneous generalized solution of degree $-\frac{1}{2}$ in t and r. We shall show that (9.5) is a generalized solution, i.e. that

$$J = \iiint_D u^\circ\left(t + \frac{t^2 - r^2}{2R},\ r,\ \varphi\right) Lv\, dt\, rdr\, d\varphi = 0$$

We introduce the notation

$$\frac{1}{2R} = a, \qquad t + \frac{t^2 - r^2}{2R} = T, \qquad \frac{v(t, r, \varphi)}{\sqrt{1 + 4a^2r^2 + 4aT}} = w(T, r, \varphi)$$

Then

$$J = J_1 + J_2$$

$$J_1 = \iiint_{D_1} u^\circ(T, r, \varphi)\left(w_{TT} - w_{rr} - \frac{1}{r}w_r - \frac{1}{r^2}w_{\varphi\varphi}\right) r\, dr\, dT d\varphi$$

$$J_2 = \iiint_{D_1} u^\circ(T, r, \varphi)\,(4Tw_{TT} + 4rw_{Tr} + 10w_T)\, r\, dr\, dT\, d\varphi$$

Since u° is a generalized solution, $J_1 = 0$. For the integral J_2, we set

$$r = \rho T,\ \ u^\circ(T, r, \varphi) = u_1(T, \rho, \varphi),\ \ w_T(T, r, \varphi) = z(T, \rho, \varphi)$$

Then

$$J_2 = \iiint_{D_2} \sqrt{T}u_1(T,\ \rho,\ \varphi)\,(4T^{5/2}z_T + 10\,T^{3/2}z)\,\rho\, d\rho\, dT\, d\varphi$$

The expression in parentheses is equal to $4\partial(T^{5/2}z)/\partial T$. Integrating by parts with respect to T and considering that $\sqrt{T}u_1(T,\rho,\varphi)$ does not depend on T (as a consequence of the homogeneity of u_1), and that in the neighborhood of the boundary $z = 0$, we obtain $J_2 = 0$.

N o t e 2. The analogous theorem is also valid for the equation

$$u_{tt} = u_{x_1x_1} + \ldots + u_{x_nx_n},$$

if u° is a homogeneous solution of degree $(1-n)/2$; here $r = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$.

We now obtain solution of the problem of diffraction of a wave due to the source. Let U be the solution (9.4) of the problem of diffraction of the plane wave (9.3). By virtue of the theorem and Note 1, the function

$$u(t, r, \varphi) = \frac{1}{2\pi\sqrt{2R}}\, U\left(t + \frac{t^2 - r^2}{2R},\ r, \varphi\right) \tag{9.6}$$

is a generalized solution. For $t < 0$

$$u\ (t,\ r,\varphi) = \frac{1}{2\pi\ \sqrt{(t+R)^2 - R^2 - r^2 + 2Rr\ \cos(\varphi-\beta)}}$$

on account of (9.3); i.e. it coincides with the wave due to the source (9.1). And so, for $t > 0$ the function (9.6) is the solution of the problem of diffraction of the wave due to the source. (The solution of this problem, expressed in another form, is known, see [7], Chapt. 5).

10. We shall now consider the three-dimensional problem of diffraction of a spherical wave due to a source by a dihedral angle (wedge) or by a cone. In particular, we shall consider a polyhedral angle with boundary conditions $u = 0$ or $\partial u/\partial n = 0$. Let r, φ, z be cylindrical coordinates. The source is assumed to act at the instant $t = -R$ at the point $r = R$, $\varphi = \beta$, $z = 0$ and the wave caused by the source to be

$$u = \frac{1}{4\pi d}\,\delta\ (t + R - d) \qquad (-R < t < 0)$$

where d is the distance from the source, δ is the delta function. The wave first reaches the obstacle (a wedge with edge $r = 0$ or a cone with vertex $r = 0$, $z = 0$) at the origin of coordinates at the instant $t = 0$. The solution u_0 of the problem of diffraction of the plane wave

$$u_0 = \frac{1}{4\pi R}\,\delta\ (t + r\cos(\varphi-\beta)) \qquad (t<0) \qquad (10.1)$$

is taken as known for the same wedge or cone. In spherical coordinates ρ, φ, ψ, where $r = \rho\ \cos\ \psi$, $z = \rho\ \sin\ \psi$, we have

$$u_0\,(t,\ r,\varphi,\ z) = u_1\,(t,\rho,\varphi,\psi) = \frac{1}{4\pi R}\,\delta\ (t + \rho\cos\psi\cos(\varphi-\beta)) \qquad (t<0) \qquad (10.2)$$

For the wedge the solution does not depend on z and may be expressed by Formula

$$u_0 = \frac{1}{4\pi R}\,\frac{\partial U^*}{\partial t}$$

where the function U^* is the same as in Section 3; i.e. U^* is the solution of the two-dimensional problem of diffraction of a plane wave by a wedge which was found by Sobolev [1]. A method of numerical solution has been indicated by Borovikov [8] for some cases of the cone.

By virtue of Note 2, the function

$$u\ (t,\ \rho,\ \varphi,\ \psi) = u_1\Big(t + \frac{t^2-\rho^2}{2R},\ \rho,\ \varphi,\ \psi\Big) \qquad (10.3)$$

is the solution of the wave equation; it obviously satisfies the boundary conditions. Because of Formula (10.2) and the properties of the δ-function

$$|a|\,\delta\ (ax) \equiv \delta\ (x), \qquad \delta\ (f\ (x)) \equiv \frac{\delta\,(x-d)}{|f'(d)|} \qquad (\text{ if } f(d)=0)$$

we have

$$u = \frac{1}{2\pi}\,\delta\,((t+R)^2 - d^2) = \frac{1}{4\pi d}\,\delta\,(t+R-d) \quad \text{for} \quad -R<t<0$$

that is, for $-R < t < 0$ the solution (10.3) coincides with the wave caused by the source. Equation (10.3) is then the solution of the problem of diffraction of a wave due to a point source by a wedge or a cone.

In the case of wedge, this solution is known [9], but in the case of the cone only the geometric acoustical expansion of the diffracted wave near the front has been available [10].

BIBLIOGRAPHY

1. Sobolev, S.L., Nekotorye voprosy teorii rasprostraneniia kolebanii (Some problems of the theory of propagation of oscillations). (Chapt. 12 in the book by F. Frank and R. Mizes, Differential and Integral Equations of Mathematical Physics), part 2, ONTI, 1937.
2. Papadopoulos, V.M., Pulse diffraction by an imperfectly reflecting wedge. J. Aust.math.Soc., Vol.2, № 1, 1961.
3. Babich, V.M. and Alekseev, A.S., O luchevom metode vychisleniia intensivnosti volnovykh frontov (A geometric acoustical method of calculating the intensity of wave fronts). Izv.Akad.Nauk SSSR, ser.geofiz., № 1, 1958.
4. Duff, G.F.D., A mixed problem for normal hyperbolic linear partial differential equations of second order. Canad.J.Math., Vol.9, № 1, 1957.
5. Filippov, A.F., O priblizhennom vychislenii otrazhennykh i prelomlennykh voln (Approximate computation of reflected and refracted waves). Izv.Akad.Nauk SSSR, ser.geofiz., № 7, 1957.
6. Borovikov, V.A., O dvumernoi zadache difraktsii na mnogougol'nike (The two-dimensional problem of diffraction by a polygon). Dokl.Akad.Nauk SSSR, Vol.144, № 4, 1962.
7. Friedlander, F.G., Zvukovye impulsy (Sound Pulses). (Cambridge University Press, 1958). IL, 1962.
8. Borovikov, V.A., O svedenii nekotorykh trekhmernykh zadach difraktsii k zadache Dirikhle dlia uravneniia Laplasa (The reduction of some three-dimensional diffraction problems to the Dirichlet problem for Laplace's equation). Dokl.Akad.Nauk SSSR, Vol.144, № 3, 1962.
9. Borovikov, V.A., O trekhmernoi zadache difraktsii na prizme (The three-dimensional problem of diffraction by a prism). Dokl.Akad Nauk SSSR, Vol.148, № 3, 1963.
10. Borovikov, V.A., O funktsii Grina dlia zadachi difraktsii na mnogogrannom ugle (The Green's function for the problem of diffraction by a polyhedral angle). Dokl.Akad.Nauk SSSR, Vol.151, № 2, 1963.

Translated by A.R.R.